Peter Lee on October 6, 2009

October-06-09
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BA $\rightsquigarrow$ Mon. Cat. on one object $\rightsquigarrow$ NAP / inverses, locality, pentagon

QTBA $\rightsquigarrow$ Braided Monoidal Category $\rightsquigarrow$ NAB / rels, also hexagons.

QTHA $\longrightarrow$ Same, with duals for objects $\rightsquigarrow$ NAT / rels.

**Objective:** Given a LBA $\mathfrak{g}$, construct a BA $H$ which is topologically free over $\mathbb{Q}[[\hbar]]$ so that

$H/\hbar H \cong \mathcal{U}(\mathfrak{g})$. [And.... ? Why not take $H = \mathcal{U}(\mathfrak{g})\hat{\otimes}\mathbb{Q}[[\hbar]]$ ?]

completed tensor product

**Strategy:** 1. Embed $\mathfrak{g}$ into a QTLBA $D\mathfrak{g}$.

2. Quantize $D\mathfrak{g}$ to get $H$.

3. Identify a subbialgebra of $H$ which is a quantization of $\mathfrak{g}$.

4. Universalize this.

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Basic Construction to retrieve $A$ from $\text{rep}(A)$:

Assoc. algebra with 1 — left A-module.

Consider $F : \text{Rep}\, A \longrightarrow \text{Vect}$, the forgetful functor:

$${}_A M \longrightarrow {}_{\mathbb{Q}}M$$

It is "representable":

<u>Prop</u> $F({}_A M) = \text{Hom}_A(A, {}_A M)$ by $\phi \in \text{Hom}_A(A, {}_A M) \longmapsto \phi(1)$

$\text{End}\, F \ni \phi$ is $\text{Rep}(A) \ni M \longmapsto \left({}_{\mathbb{Q}}M \xrightarrow{\phi_M} {}_{\mathbb{Q}}M\right)$

s.t.

$$\begin{array}{ccc} {}_{\mathbb{Q}}M & \xrightarrow{\phi_M} & {}_{\mathbb{Q}}M \\ F(\psi)\downarrow & & \downarrow F(\psi) \\ {}_{\mathbb{Q}}N & \longrightarrow & {}_{\mathbb{Q}}N \end{array}$$

whenever $\psi : M \to N$ in $\text{Rep}(A)$

$$\otimes N \longrightarrow \otimes N \qquad \text{in } Rep(A)$$

<u>Lemma</u> $A \cong End(F)$

<u>Pf</u> Construct $\theta : A \xrightarrow{\sim} End(F)$ by

$$a \in A \mapsto \theta(a)_M := \text{left multiply by } a.$$

Injective: assume $\theta(a)_A = \theta(a')_A$, then

$$a = a \cdot 1 = a' \cdot 1 = a'$$

Surjective: Suppose $\phi \in End(F)$. Let $a = \phi_A(1)$;

claim: $\theta(a) = \phi \ldots$

Order of proceedings: First on $A$,
then on free $A$-modules like $A \otimes V$,
then on arbitrary $A$-modules.

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A nice property for a functor $F : Rep\ A \longrightarrow Vect$ would be $F(M \otimes N) = F(M) \otimes F(N)$.

We relax this to

$$F(M \otimes N) \xrightarrow{\sim} F(M) \otimes F(N)$$

<u>Def</u> A tensor structure on $F : \mathcal{C} \to \mathcal{D}$ (monoidal categories)
(or a "monoidal functor")

is a natural collection $J_{M,N} : F(M) \otimes F(N) \longrightarrow F(M \otimes N)$
for $M, N \in Obj(\mathcal{C})$, as well as

$$j : 1_{\mathcal{D}} \xrightarrow{\sim} F(1_{\mathcal{C}}), \quad \text{s.t.}$$

$$\begin{tikzcd}
1 \otimes F(M) \arrow[r, "l_{\mathcal{D}}"] \arrow[d, "j \otimes 1"'] & F(M) \\
F(1) \otimes F(M) \arrow[r, "J_{1,M}"] & F(1 \otimes M) \arrow[u, "F(l_{\mathcal{C}})"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(F(M) \otimes F(N)) \otimes F(P) \arrow[r, "J \otimes 1"] \arrow[d, "\Phi_{\mathcal{D}}"] & F(M \otimes N) \otimes F(P) \arrow[r, "J"] & F((M \otimes N) \otimes P) \arrow[d, "F(\Phi_{\mathcal{C}})"] \\
F(M) \otimes (F(N) \otimes F(P)) \arrow[r, "1 \otimes J"] & F(M) \otimes F(N \otimes P) \arrow[r, "J"] & F(M \otimes (N \otimes P))
\end{tikzcd}$$

$$F(M)\otimes(F(N)\otimes F(P)) \xrightarrow{1\otimes J} F(M)\otimes F(N\otimes P) \xrightarrow{J} F(M\otimes(N\otimes P))$$

<u>Tanaka-Krein Duality for Finite groups</u> Let $G$ be a finite group, $A = \mathbb{Q}[G]$ the group ring, $\mathcal{C} = Rep(A)$; it is a monoidal category.

Consider $F : Rep\ A \longrightarrow Vect$. It has a tensor structure $J$:

the forgetful functor

$$J_{M,N} : {}_{\mathbb{Q}}M \otimes {}_{\mathbb{Q}}N \longrightarrow {}_{\mathbb{Q}}(M\otimes N)$$ — Just take the identity!

Consider $$Aut_{\otimes} F \subset Aut(F)$$

$$Aut_{\otimes} F = \left\{ \gamma \in End\ F \text{ invertible} : \begin{tikzcd} {}_{\mathbb{Q}}M \otimes {}_{\mathbb{Q}}N \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "J"] \arrow[dr, phantom, "\circlearrowright"] & {}_{\mathbb{Q}}(M\otimes N) \arrow[d, "J"] \\ {}_{\mathbb{Q}}(M\otimes N) \arrow[r, "\gamma"] & {}_{\mathbb{Q}}(M\otimes N) \end{tikzcd} \right\}$$

<u>Claim</u> $G \cong Aut_{\otimes} F$

We already know that $End F \cong A = \mathbb{Q}[G]$.

Suppose $a \in Aut_{\otimes} F$ ... the rest follows from $G = \{h \in A : \Delta(h) = h\otimes h\}$:

$$a(m\otimes n) = \Delta(a)(m\otimes n) = (a\otimes a)(m\otimes n)$$